Next aim: (1)

Dirichlet's density thm:

$a, N \geq 1$, $(a,N)=1$

$\Rightarrow \exists$ inf. primes p, s.t. $p \equiv a \ (N)$

In fact,

$$\sum_{\substack{p \text{ prime} \\ p \equiv a \ (N)}} \frac{1}{p^s} \underset{s \searrow 1}{\sim} \frac{1}{\varphi(N)} \log\left(\frac{1}{s-1}\right)$$

("$\{p \text{ prime}, p \equiv a(N)\}$ has Dirichlet density $\frac{1}{\varphi(N)}$")

Here: $\varphi(N) = \#(\mathbb{Z}/N)^*$

Recall: 1) $K = \mathbb{Q}(\zeta_N)$

$$\Rightarrow \zeta_K(s) = \prod L(\chi, s)$$ L-fct. ass. to primitive repr. of χ

$$\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$$

has simple pole at $s=1$

residue $\dfrac{2^{r_1}(2\pi)^{r_2} R_N \cdot h}{w\sqrt{|\Delta_N|}} \neq 0$

2) χ non-trivial

$\Rightarrow L(\chi, s)$ holomorphic

to $\operatorname{Re} s > 0$

In part, $L(\chi, 1) \neq 0$ for χ non-trivial

Now start proof of Dirichlet's thm: ③

$$\sum_{\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times} \chi(\bar{a}^{-1}) \cdot \log L(\chi, s)$$

$$= \sum_{\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}} \chi(\bar{a}^{-1}) \cdot \sum_p \log\left(\frac{1}{1-\chi(p)p^{-s}}\right)$$

$$= \sum_\chi \chi(\bar{a}^{-1}) \sum_p \sum_{m \geq 1} \frac{\chi(p^m)}{m \cdot p^{ms}} = (*)$$

Evaluate $\sum_\chi \chi(\bar{a}^{-1} p^m)$

Lemma: H finite, ab., $h \in H$

$$\Rightarrow \sum_{\chi : H \to \mathbb{C}^\times} \chi(h) = \begin{cases} |H| , & h = 1 \\ 0 , & h \neq 1 \end{cases}$$

Proof: ~~Über H~~

Recall $\hat{H} = \mathrm{Hom}(H, \mathbb{C}^\times)$

$\Rightarrow H \xrightarrow{\simeq} \hat{\hat{H}}$, $g \mapsto (\chi \mapsto \chi(g))$

(Reduce to $H = \mathbb{Z}/n \Rightarrow \hat{H} \simeq \mu_n \simeq \mathbb{Z}/n$)

and $$\sum_{h \in H} \chi(h) = \begin{cases} |H| , & \chi = 1 \\ 0 , & \chi \neq 1 \end{cases}$$

$\Rightarrow$ Apply this for $\hat{H}$ $\square$

$$\Rightarrow (*) = \varphi(N)\cdot \sum_{\substack{p,\, m\geq 1 \\ p^m \equiv a (N)}} \frac{1}{m p^{ms}}$$

$$= \varphi(N)\cdot \sum_{\substack{p \in P \\ p \equiv a (N)}} \frac{1}{p^s} + \underbrace{\varphi(N)\cdot \sum_{\substack{p,\, m\geq 2 \\ p^m \equiv a (N)}} \frac{1}{m\cdot p^{ms}}}_{}$$

remains bdd for

$s \to 1$

$$\sum_{n=1}^{\infty} \frac{1}{(n^2)^s} < \infty$$

$$\| \\ \zeta(2)$$

On the other hand

$$\sum_{\chi} \chi(a^{-1}) \cdot \log L(\chi,s)$$

$$= \log \zeta(s) + \sum_{\chi \neq 1} \chi(a^{-1}) \log(L(\chi,s))$$

$\downarrow$ $L(\chi,1) \neq 0$

remains bdd, when $s \searrow 1$

$$\Rightarrow \log \frac{1}{s-1} \sim \log \zeta(s)$$

$$\sim \varphi(N) \cdot \sum_{p \equiv a (N)} \frac{1}{p^s} \quad \text{for } s \searrow 1$$

$$\Rightarrow \sum_{p \equiv a(N)} \frac{1}{p^s} \sim \frac{1}{\varphi(N)} \cdot \log \frac{1}{s-1} \qquad \square$$

More fun with L-functions

K biquadratic field, i.e.

$K/\mathbb{Q}$ Galois with grp
$$G \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$$

(note $K \subseteq \mathbb{Q}(\zeta_N)$ for some $N \geq 1$)

$\Rightarrow$
$$\begin{tikzcd}
 & K \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
K_1 \arrow[dr, dash] & K_2 \arrow[d, dash] & K_3 \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

three quadratic subfields

$(K_i \subseteq \mathbb{Q}(\zeta_{N_i}))$

$\Rightarrow K_i \cdot K_j \subseteq \mathbb{Q}(\zeta_{N_i}) \mathbb{Q}(\zeta_{N_j}) \subseteq \mathbb{Q}(\zeta_{N_i \cdot N_j})$

Let χ_i (prim.) Dirichlet characters ass. to K_i

$$\frac{2^{r_1,K}(2\pi)^{s,K} R_K \cdot h_K}{w_K \cdot \sqrt{|\Delta_K|}}$$

$\zeta_K(s)$ factorizes

$$= L(\chi_1,1) \cdot L(\chi_2,1) \cdot L(\chi_3,1)$$

$$= \frac{2^{r_1,K_1}(2\pi)^{s,K_1} R_{K_1} \cdot h_{K_1}}{w_{K_1} \cdot \sqrt{|\Delta_{K_1}|}} \cdot$$

. . . . .

$\Rightarrow$ Can try to calculate h_K from $h_{K_1}, h_{K_2}, h_{K_3}$

General case: Fröhlich / Taylor ⑨
"Algebraic number theory" (e.g. Thm 74)

Let's consider

$K_1 = \mathbb{Q}(\sqrt{p})$, p prime$_{\geq 0}$, $p \equiv 1 (4)$

($\rightsquigarrow \Delta_{K_1} = p$)

$K_2 = \mathbb{Q}(\sqrt{-q})$, q prime$_{\geq 0}$, $\equiv 3 (4)$

($\rightsquigarrow \Delta_{K_2} = -q$)

$q \geq 7$ ($\rightsquigarrow w_{K_2} = 2$)

$K_3 = \mathbb{Q}(\sqrt{-p \cdot q})$

Get

	K_1	K_2	K_3	K
r_1	2	0	0	0
r_2	0	1	1	2
$\lvert\Delta\rvert$	p	$+q$	$p\cdot q$	$p^2\cdot q^2$
w	2	2	2	2

$$\Rightarrow \frac{(\cancel{2}\pi)^2\cdot R_K\cdot h_K}{\cancel{2}\cdot\sqrt{\cancel{p}\cdot q}}$$

$$= \frac{\cancel{2}^2\cdot R_{K_1}\cdot h_{K_1}}{\cancel{2}\cdot\sqrt{p}}\cdot\frac{\cancel{2\pi}\cdot h_{K_2}}{\cancel{2}\cdot\sqrt{q}}\cdot\frac{\cancel{2\pi}\, h_{K_3}}{\cancel{2}\cdot\sqrt{\cancel{p}q}}$$

$$\Rightarrow R_K\cdot h_K = R_{K_1}\cdot h_{K_1}\cdot h_{K_2}\cdot h_{K_3}$$

Claim: $R_K = 2 R_{K_1}$ $\left(\Leftrightarrow h_K = \frac{1}{2} h_{K_1} \cdot h_{K_2} \cdot h_{K_3}\right)$

STP: $U_K = U_{K_1}$

Indeed, If $\langle \varepsilon \rangle = U_{K_1}/\{\pm 1\} = U_K/\{\pm 1\}$

$\sigma_1, \sigma_2 : K_1 \to \mathbb{R}$

$\tau_1, \tau_2 : K \to \mathbb{C}$, $\tau_i|_{K_1} = \sigma_i$

$$R_K = \left|\det \begin{pmatrix} \frac{1}{2} & 2\log|\tau_1(\varepsilon)| \\ \frac{1}{2} & 2\log|\tau_2(\varepsilon)| \end{pmatrix}\right|$$

$$= \left|\det \begin{pmatrix} \frac{1}{2} & 2\cdot\log|\sigma_1(\varepsilon)| \\ \frac{1}{2} & 2\cdot\log|\sigma_2(\varepsilon)| \end{pmatrix}\right| = 2 R_{K_1}$$

La: $\varepsilon \in U_K \Rightarrow \frac{\varepsilon}{\bar{\varepsilon}} \in W_K$

Proof: Holds more gen. $\frac{\varepsilon}{\sigma(\varepsilon)} \in W_K$

$$\begin{array}{ll} F & \rightsquigarrow \text{tot. imag. } (r_1 = 0) \\ \Big| & \text{quadratic} \qquad \text{"}F \text{ CM-field"} \\ F_0 & \rightsquigarrow \text{totally real } (r_2 = 0) \end{array}$$

$\Rightarrow$ Let $\sigma \in Gal(F/F_0)$

$\Rightarrow$ If $\tau : F \to \mathbb{C}$ any $\mathbb{Q}$-emb.

$\Rightarrow$ $\bar{\tau} = \tau \circ \sigma$

(If F Galois $\Rightarrow \sigma \in Gal(F/\mathbb{Q})$ is central

Set $\bar{\varepsilon} = \sigma(\varepsilon)$)

Pick $\varepsilon \in U_F$, $\tau : F \to \mathbb{C}$

$$\Rightarrow \left|\tau\left(\frac{\varepsilon}{\sigma(\varepsilon)}\right)\right| = \left|\frac{\tau(\varepsilon)}{\overline{\tau(\varepsilon)}}\right| = 1$$

$$\Rightarrow \frac{\varepsilon}{\sigma(\varepsilon)} \in W_K$$

$\square$

Moreover,

$$1 \to U_{F_0} \cdot W_F \to U_F \to W_F / W_F^{2}$$

$$\varepsilon \mapsto \frac{\varepsilon}{\sigma(\varepsilon)}$$

exact.

Indeed, pick $\varepsilon \in U_F$, s.t. $\frac{\varepsilon}{\sigma(\varepsilon)} = \zeta^2$, $\zeta \in W_F$

$\Rightarrow \sigma(\varepsilon \cdot \zeta^{-1}) = \sigma(\varepsilon) \cdot \zeta$

$= \varepsilon \cdot \zeta^{-2} \cdot \zeta = \varepsilon \cdot \zeta^{-1}$

$\Rightarrow \varepsilon \cdot \zeta^{-1} \in U_{F_0}$

$\Rightarrow \varepsilon = (\varepsilon \cdot \zeta^{-1}) \cdot \zeta \in U_{F_0} \cdot W_F$.

In our situation, furthermore

$U_{K_1} = U_{K_1} \cdot W_K \subseteq U_K$

Assume $U_K \neq U_{K_1}$

$\Rightarrow \exists\, u \in U_K \setminus U_{K_1}$, s.t. $u^2 \in U_{K_1}$

$\uparrow$

$W_K / W_K^2 = \{\pm 1\}$

$\Rightarrow \quad K = K_1(\sqrt{u^2})$

$\Rightarrow$ only primes above 2 ramify in K/K_1

↯ as q ramifies K/K_1

$\mathrm{Disk}(1, u) = \pm 4 \cdot \underbrace{N_{K/K_1}(u)}_{\text{unit}}$

Upshot: $R_K = 2 \cdot R_{K_1}$

$\Rightarrow \quad h_K = \frac{1}{2} h_{K_1} \cdot h_{K_2} \cdot h_{K_3}$ (Yeah!)